Electronic Journal of Qualitative Theory of Differential Equations
2017, No. **57**, 1–17; doi: 10.14232/ejqtde.2017.1.57 http://www.math.u-szeged.hu/ejqtde/

Fractional Laplacian system involving doubly critical nonlinearities in $\mathbb{R}^N$

Li Wang[1], **Binlin Zhang**[✉ 2] and **Haijin Zhang**[1]

[1]College of Science, East China Jiaotong University, Nanchang, 330013, P.R. China
[2]Department of Mathematics, Heilongjiang Institute of Technology, Harbin, 150050, P.R. China

Received 15 March 2017, appeared 20 July 2017
Communicated by Patrizia Pucci

Abstract. In this article, we are interested in a fractional Laplacian system in $\mathbb{R}^N$, which involves critical Sobolev-type nonlinearities and critical Hardy–Sobolev-type nonlinearities. By using variational methods, we investigate the extremals of the corresponding best fractional Hardy–Sobolev constant and establish the existence of solutions. To our best knowledge, our main results are new in the study of the fractional Laplacian system.

Keywords: fractional Laplacian system, doubly critical nonlinearities, variational methods.

2010 Mathematics Subject Classification: 35A15, 35R11, 35J50.

1 Introduction and main result

In this article, we are concerned with the existence of solutions for the following fractional Laplacian system in $\mathbb{R}^N$:

$$\begin{cases} (-\Delta)^s u - \mu \dfrac{u}{|x|^{2s}} = (\mathcal{I}_a * |u|^{2^\sharp_{h,a}})|u|^{2^\sharp_{h,a}-2}u + \dfrac{|u|^{2^*_{s,b}-2}u}{|x|^b} + \dfrac{\eta\alpha}{\alpha+\beta}\dfrac{|u|^{\alpha-2}u|v|^\beta}{|x|^b}, \\ (-\Delta)^s v - \mu \dfrac{v}{|x|^{2s}} = (\mathcal{I}_a * |v|^{2^\sharp_{h,a}})|v|^{2^\sharp_{h,a}-2}v + \dfrac{|v|^{2^*_{s,b}-2}v}{|x|^b} + \dfrac{\eta\beta}{\alpha+\beta}\dfrac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^b}, \end{cases} \tag{1.1}$$

where $\mathcal{I}_a(x) = \frac{\Gamma(\frac{N-2}{2})}{2^a\pi^{N/2}\Gamma(\frac{a}{2})|x|^{N-a}}$ is a Riesz potential, for simplicity, we set $\mathcal{I}_a(x) = \frac{1}{|x|^{N-a}}$, $\mu, \eta \in \mathbb{R}^+_0$, $0 < a, b < 2s < N$, $\alpha > 1$, $\beta > 1$, $\alpha + \beta = 2^*_{s,b} = \frac{2(N-b)}{N-2s}$ and $2^\sharp_{h,a} = \frac{N+a}{N-2s}$ are fractional critical exponents for Sobolev-type embeddings. The fractional Laplace operator $(-\Delta)^s$ is defined by

$$(-\Delta)^s = \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u(\xi)) \quad \text{for all } u \in C_0^\infty(\mathbb{R}^N),\ \xi \in \mathbb{R}^N,$$

✉ Corresponding author. Emails: wangli.423@163.com (L. Wang), zhangbinlin2012@163.com (B. Zhang), 1015330036@qq.com (H. Zhang)

where $\mathcal{F}u$ denotes the Fourier transform of u. Weak solutions of (1.1) will be found in the space $H = \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$, where $\dot{H}^s(\mathbb{R}^N)$ is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\xi|^{2s}|\hat{u}(\xi)|^2 \mathrm{d}\xi. \tag{1.2}$$

Therefore, for $s > 0$, we have

$$\|(-\Delta)^{\frac{s}{2}} u\|^2_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\xi|^{2s}|\hat{u}(\xi)|^2 \mathrm{d}\xi. \tag{1.3}$$

By a (weak) solution (u, v) of problem (1.1), we mean that $(u, v) \in H$ satisfies

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left[(-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \phi_1 + (-\Delta)^{\frac{s}{2}} v (-\Delta)^{\frac{s}{2}} \phi_2 - \mu \left(\frac{u\phi_1 + v\phi_2}{|x|^{2s}} \right) \right] \mathrm{d}x \\
&\quad = \iint_{\mathbb{R}^{2N}} \frac{|u(y)|^{2^\sharp_{h,a}} |u(x)|^{2^\sharp_{h,a}-2} u(x)\phi_1(x) + |v(y)|^{2^\sharp_{h,a}} |v(x)|^{2^\sharp_{h,a}-2} v(x)\phi_2(x)}{|x-y|^{N-a}} \mathrm{d}x\mathrm{d}y \\
&\qquad + \int_{\mathbb{R}^N} \frac{|u|^{2^*_{s,b}-2} u\phi_1 + |v|^{2^*_{s,b}-2} v\phi_2 + \eta(\alpha |u|^{\alpha-2} u\phi_1 |v|^\beta + \beta |u|^\alpha |v|^{\beta-2} v\phi_2)}{|x|^b} \mathrm{d}x
\end{aligned}$$

for all $\phi_1, \phi_2 \in \dot{H}^s(\mathbb{R}^N)$.

In recent years, much attention has been paid to fractional and non-local operators. More precisely, this type of operators arises in a quite natural way in many different applications, such as, finance, physics, fluid dynamics, population dynamics, image processing, minimal surfaces and game theory, see [4, 11] and the references therein. In particular, there are some remarkable mathematical models involving the fractional Laplacian, such as, the fractional Schrödinger equation (see [18, 31]), the fractional Kirchhoff equation (see [1, 13, 24, 25]), the fractional porous medium equation (see [5]) and so on.

Problems with one nonlinearity or two nonlinearities involving the Laplacian and the fractional Laplacian have been studied by many authors. For example, we refer, in bounded domains to [14, 20, 21, 27, 28, 30], while in the entire space to [12, 16, 22]. In [8], Filippucci, Pucci and Robert proved that there exists a positive solution for a p-Laplacian problem with critical Sobolev and Hardy–Sobolev terms. In [15], Fiscella, Pucci and Saldi dealt with the existence of nontrivial nonnegative solutions of Schrödinger–Hardy systems driven by two possibly different fractional $\wp$-Laplacian operators, also including critical nonlinear terms, where the nonlinearities do not necessarily satisfy the Ambrosetti–Rabinowitz condition. It is natural to consider the concentration–compactness principle for critical problems. However, due to the nonlocal feature of the fractional Laplacian, it is difficult to use the concentration–compactness principle directly, since one needs to estimate commutators of the fractional Laplacian and smooth functions. A natural strategy, which is named by the s-harmonic extension, is to transform the nonlocal problem into a local problem, as Caffarelli and Silvestre performed in [3]. Since that, many interesting results in the classical elliptic problems have been extended to the setting of the fractional Laplacian. For example, Ghoussoub and Shakerian in [16] combined the s-harmonic extension with the concentration-compactness principle to investigate the existence of solutions for a doubly critical problem involving the fractional Laplacian. It is worthy pointing out that Yang and Wu in [33] showed the existence of solutions for problem (1.1) with $\eta = 0$, by using the elementary approach without the use of the concentration-compactness principle or the extension argument of Caffarelli and Silvestre in [3].

In the doubly critical case, two critical nonlinearities interact to each other. There is an asymptotic competition between the energy carried by the two critical nonlinearities. Obviously, the combination of the two critical exponents induces more difficulties. When one critical exponent is only involved, there are solutions to the corresponding equations: in general, these solutions are radially symmetric with respect to the origin of the domain and are explicit, see for instance [23] for the details. However, very few information have been known in our setting, especially for system, here we just refer the reader to an interesting literature [12].

In this paper, we are interested in the existence of solutions for system (1.1) involving doubly critical exponents, by using a refinement of the Sobolev inequality which is related to the Morrey space. A measurable function $u : \mathbb{R}^N \to \mathbb{R}$ belongs to the Morrey space $\mathcal{L}^{p,\gamma}(\mathbb{R}^N)$ with $p \in [1, +\infty)$ and $\gamma \in (0, N]$, if and only if

$$\|u\|^p_{\mathcal{L}^{p,\gamma}(\mathbb{R}^N)} = \sup_{R>0, x\in\mathbb{R}^N} R^{\gamma-N} \int_{B_R(x)} |u(y)|^p \mathrm{d}y < \infty. \tag{1.4}$$

By the Hölder inequality, we can verify that $L^{2^*_s}(\mathbb{R}^N) \hookrightarrow \mathcal{L}^{p,\frac{N-2s}{2}p}(\mathbb{R}^N)$ for $1 \le p < 2^*_s = \frac{2N}{N-2s}$, and for $1 < q < p < 2^*_s$ we have

$$\mathcal{L}^{p,\frac{N-2s}{2}p}(\mathbb{R}^N) \hookrightarrow \mathcal{L}^{q,\frac{N-2s}{2}q}(\mathbb{R}^N).$$

Moreover, here holds $\mathcal{L}^{p,\gamma}(\mathbb{R}^N) \hookrightarrow \mathcal{L}^{1,\frac{\gamma}{p}}(\mathbb{R}^N)$ provided that $p \in (1, +\infty)$ and $\gamma \in (0, N)$.

The following refinement of Hardy–Sobolev inequalities were proved in [19] and [32].

Proposition 1.1. ([32, Theorem 1.1]). *For any $0 < b < 2s < N$, there exists $C > 0$ such that for θ and r satisfying $\max\{\frac{N-2s}{N-b}, \frac{2s-b}{N-b}\} \le \theta < 1 \le r < 2^*_{s,b}$, there holds*

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{s,b}}}{|x|^b} \mathrm{d}x\right)^{\frac{1}{2^*_{s,b}}} \le C\|u\|^\theta_{\dot{H}^s(\mathbb{R}^N)} \|u\|^{1-\theta}_{\mathcal{L}^{r,\frac{r(N-2s)}{2}}(\mathbb{R}^N)} \tag{1.5}$$

for any $u \in \dot{H}^s(\mathbb{R}^N)$.

In the present paper, we work in the product space $H = \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N)$ be the Cartesian product of two Hilbert spaces, which is a reflexive Banach space endowed with the norm

$$\|(u,v)\|^2 = \|(u,v)\|^2_H = \|u\|^2_{\dot{H}^s(\mathbb{R}^N)} + \|v\|^2_{\dot{H}^s(\mathbb{R}^N)},$$

where

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}} u|^2 - \mu \frac{|u|^2}{|x|^{2s}}\right) \mathrm{d}x.$$

Solutions of (1.1) are equivalent to a nonzero critical points of the functional

$$I(u,v) = \frac{1}{2}\|(u,v)\|^2 - \frac{1}{2\cdot 2^\sharp_{h,a}} \iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{2^\sharp_{h,a}}|u(y)|^{2^\sharp_{h,a}} + |v(x)|^{2^\sharp_{h,a}}|v(y)|^{2^\sharp_{h,a}}}{|x-y|^{N-a}} \mathrm{d}x\mathrm{d}y$$
$$- \frac{1}{2^*_{s,b}} \int_{\mathbb{R}^N} \frac{|u|^{2^*_{s,b}} + |v|^{2^*_{s,b}} + \eta|u|^\alpha|v|^\beta}{|x|^b} \mathrm{d}x,$$

which is defined on H, and $I \in C^1(H, \mathbb{R})$. We say a pair of functions $(u,v) \in H$ is called to be a solution of (1.1) if

$$u \ne 0,\ v \ne 0, \qquad \langle I'(u,v), (\phi_1, \phi_2)\rangle = 0, \qquad \forall (\phi_1, \phi_2) \in H. \tag{1.6}$$

If $(u,v)=(u,0)$ or $(u,v)=(0,v)$, we say that they are the semi-nontrivial solution. In this case, system can be seen as a singular equation, that is $\eta=0$, see [33] for the details.

The main result of this paper can be concluded in the following theorem.

Theorem 1.2. *If $0\le\mu<\mu_*=4^s\frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}$, then problem (1.1) possesses at least one nontrivial solution in H.*

Remark 1.3. To the best of our knowledge, Theorem 1.2 is new in the study of the fractional Laplacian system involving doubly critical nonlinearities in the whole space. We mainly follow the idea of [33] to prove our main result.

This paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, the extremals of the corresponding best fractional Hardy–Sobolev constant are achieved. In Section 4, we give the proof of Theorem 1.2.

Throughout this paper, we will use the following notations: $tz:=t(u,v)=(tu,tv)$ for all $(u,v)\in H$ and $t\in\mathbb{R}$; (u,v) is said to be nonnegative in $\mathbb{R}^N$ if $u\ge0$ and $v\ge0$ in $\mathbb{R}^N$; (u,v) is said to be positive in $\mathbb{R}^N$ if $u>0$ and $v>0$ in $\mathbb{R}^N$; $B_r(0)=\{x\in\mathbb{R}^N:\ |x|<r\}$ is a ball in $\mathbb{R}^N$ of radius $r>0$ at the origin; $o(1)$ is a generic infinitesimal value. We always denote positive constants as C for convenience.

2 Preliminaries

In this section, we recall the fractional Sobolev inequality. For $N>2s$, the fractional Sobolev embedding $\dot H^s(\mathbb{R}^N)\hookrightarrow L^{\frac{2N}{N-2s}}(\mathbb{R}^N)$ was considered in [6,7]. The continuity of this inclusion corresponds to the inequality

$$\|u\|^2_{2_s^*(\mathbb{R}^N)}\le S_\mu\|u\|^2_{\dot H^s(\mathbb{R}^N)}. \tag{2.1}$$

The best constant S_μ in (2.1) was computed (see Theorem 1.1 in [7]). Using the moving plane method for integral equations, Chen, Li and Ou in [6] classified the solutions of an integral equation, which is related to the problem

$$(-\Delta)^s u=|u|^{2_s^*-2}u\quad\text{in }\mathbb{R}^N. \tag{2.2}$$

The positive regular solutions of (2.2), which verify the equality in (2.1), are precisely given by

$$U(x)=\frac{1}{(\lambda^2+|x-x_0|^2)^{\frac{N-2s}{2}}} \tag{2.3}$$

for $\lambda>0$ and $x_0\in\mathbb{R}^N$.

On the other hand, the Hardy–Littlewood–Sobolev inequality yields

$$\left(\iint_{\mathbb{R}^{2N}}\frac{|u(x)|^{2^\sharp_{h,a}}|u(y)|^{2^\sharp_{h,a}}}{|x-y|^{N-a}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2^\sharp_{h,a}}}\le\|u\|^2_{\frac{2N}{N-2s}}\le C\|(-\Delta)^{\frac{s}{2}}u\|_2^2\,, \tag{2.4}$$

and the equality in (2.4) holds if and only if u is given by (2.3). Thus, the exponent $2^\sharp_{h,a}$ is critical in the sense that it is the limit exponent for the Sobolev-type inequality (2.4). Taking into account Proposition 1.1 and (2.4), we obtain the following inequality:

$$\left(\iint_{\mathbb{R}^{2N}}\frac{|u(x)|^{2^\sharp_{h,a}}|u(y)|^{2^\sharp_{h,a}}}{|x-y|^{N-a}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2^\sharp_{h,a}}}\le C\|u\|^{2\theta}_{\dot H^s}\|u\|^{2(1-\theta)}_{\mathcal{L}^{2,N-2s}} \tag{2.5}$$

for $u\in\dot H^s(\mathbb{R}^N)$.

3 Minimizers of $S_{s,b}$

In this section, we show that the best constant $S_{s,b}$ in our context can be achieved. Moreover, we investigate the intrinsic relation between $S_{s,b}$ and the best fractional Hardy–Sobolev constant with single equation.

For $\mu > 0$, we define

$$\mu_* = \inf\left\{ \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x}{\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2s}} \mathrm{d}x},\ u \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\} \right\}. \tag{3.1}$$

Here we remark that μ_* in (3.1) was showed in [34] that

$$\mu_* = 4^s \frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}.$$

Evidently, from (3.1) we have the fractional Hardy–Sobolev inequality

$$\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2s}} \mathrm{d}x \le \mu_*^{-1} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x. \tag{3.2}$$

If $0 \le \mu < \mu_*$, by the fractional Sobolev inequality

$$\left(\int_{\mathbb{R}^N} |u|^{2_s^*} \mathrm{d}x \right)^{\frac{2}{2_s^*}} \le S^{-1} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x \tag{3.3}$$

and (3.2), we have

$$S\left(1 - \frac{\mu}{\mu_*}\right) \left(\int_{\mathbb{R}^N} |u|^{2_s^*} \mathrm{d}x \right)^{\frac{2}{2_s^*}} \le \int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}} u|^2 - \mu \frac{|u|^2}{|x|^{2s}} \right) \mathrm{d}x. \tag{3.4}$$

Then, for $0 \le \mu < \mu_*$, we define the functional

$$I_{s,a}(u,v) = \frac{\int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}} u|^2 - \mu \frac{|u|^2}{|x|^{2s}} + |(-\Delta)^{\frac{s}{2}} v|^2 - \mu \frac{|v|^2}{|x|^{2s}} \right) \mathrm{d}x}{\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{2^\sharp_{h,a}} |u(y)|^{2^\sharp_{h,a}} + |v(x)|^{2^\sharp_{h,a}} |v(y)|^{2^\sharp_{h,a}}}{|x-y|^{N-a}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2^\sharp_{h,a}}}} \tag{3.5}$$

and

$$I_{s,b}(u,v) = \frac{\int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}} u|^2 - \mu \frac{|u|^2}{|x|^{2s}} + |(-\Delta)^{\frac{s}{2}} v|^2 - \mu \frac{|v|^2}{|x|^{2s}} \right) \mathrm{d}x}{\left(\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_{s,b}} + |v(x)|^{2^*_{s,b}} + \eta |u(x)|^{\alpha} |v(x)|^{\beta}}{|x|^b} \mathrm{d}x \right)^{\frac{2}{2^*_{s,b}}}}. \tag{3.6}$$

Consider the minimization problem

$$S_{s,a} = \inf\Big\{ I_{s,a}(u,v) : u,v \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\} \Big\}, \tag{3.7}$$

and

$$S_{s,b} = \inf\Big\{ I_{s,b}(u,v) : u,v \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\} \Big\}. \tag{3.8}$$

The following result shows that for $0 \le \mu < \mu_*$, $S_{s,a}, S_{s,b}$ are achieved.

Lemma 3.1. *If $0 \le \mu < \mu_*$, then $S_{s,a}$ and $S_{s,b}$ are achieved respectively by a pair of radially symmetric and nonnegative functions.*

Proof. Here we only give the proof of process for $S_{s,a}$ being achieved. With minor changes, we can also get that $S_{s,b}$ is achieved by a pair of radially symmetric nonnegative functions.

Let $\{(u_n, v_n)\}_n$ be a minimizing sequence of $S_{s,a}$, that is

$$\int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} u_n|^2 - \mu \frac{|u_n|^2}{|x|^{2s}} + |(-\Delta)^{\frac{s}{2}} v_n|^2 - \mu \frac{|v_n|^2}{|x|^{2s}} \right] \mathrm{d}x \to S_{s,a}$$

as $n \to \infty$ and

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)|^{2^\sharp_{h,a}} |u_n(y)|^{2^\sharp_{h,a}} + |v_n(x)|^{2^\sharp_{h,a}} |v_n(y)|^{2^\sharp_{h,a}}}{|x-y|^{N-a}} \mathrm{d}x\mathrm{d}y = 1.$$

Inequality in (2.5) enables us to find $C > 0$ such that

$$\|u_n\|_{\mathcal{L}^{2,N-2s}(\mathbb{R}^N)} \ge C,$$

and the Sobolev embedding $\dot{H}^s(\mathbb{R}^N) \hookrightarrow \mathcal{L}^{2,N-2s}(\mathbb{R}^N)$ gives

$$\|u_n\|^2_{\mathcal{L}^{2,N-2s}(\mathbb{R}^N)} \le C.$$

So we may find $\lambda_n > 0$ and $x_n \in \mathbb{R}^N$ such that

$$\lambda_n^{-2s} \int_{B_{\lambda_n}(x_n)} |u_n|^2 \mathrm{d}y \ge \|u_n\|^2_{\mathcal{L}^{2,N-2s}(\mathbb{R}^N)} - \frac{C}{2n} \ge C_1 > 0.$$

Let $\tilde{u}_n(x) = \lambda_n^{\frac{N-2s}{2}} u_n(\lambda_n x)$. Then

$$\lambda_n^{-2s} \int_{B_1(\frac{x_n}{\lambda_n})} |\tilde{u}_n|^2 \mathrm{d}y \ge C_1 > 0.$$

Similarly, we can get that

$$\lambda_n^{-2s} \int_{B_1(\frac{x_n}{\lambda_n})} |\tilde{v}_n|^2 \mathrm{d}y \ge C_1 > 0,$$

where $\tilde{v}_n(x) = \lambda_n^{\frac{N-2s}{2}} v_n(\lambda_n x)$.

By simple computation, we can get $I(u_n, v_n) = I(\tilde{u}_n(x), \tilde{v}_n(x))$, and then $\{(\tilde{u}_n(x), \tilde{v}_n(x))\}_n$ is also a minimizing sequence of $S_{s,a}$. We can also show that $\{(\tilde{u}_n(x), \tilde{v}_n(x))\}_n$ is bounded in H. Hence, we may assume

$$\begin{aligned} (\tilde{u}_n(x), \tilde{v}_n(x)) &\rightharpoonup (\tilde{u}(x), \tilde{v}(x)) && \text{weakly in } \dot{H}^s(\mathbb{R}^N) \times \dot{H}^s(\mathbb{R}^N), \\ (\tilde{u}_n(x), \tilde{v}_n(x)) &\rightharpoonup (\tilde{u}(x), \tilde{v}(x)) && \text{weakly in } \left(L^p_{\text{loc}}(\mathbb{R}^N)\right)^2 \text{ for all } 1 \le p < 2^*_s, \\ (\tilde{u}_n(x), \tilde{v}_n(x)) &\to (\tilde{u}(x), \tilde{v}(x)) && \text{a.e. in } \mathbb{R}^N \times \mathbb{R}^N. \end{aligned}$$

We claim that $\{\frac{x_n}{\lambda_n}\}_n$ is uniformly bounded in n. Indeed, for any $0 < \kappa < 2s$, we observe, by the Hölder inequality, that

$$\begin{aligned} 0 < C_1 &\le \int_{B_1(\frac{x_n}{\lambda_n})} |\tilde{u}_n|^2 \mathrm{d}y = \int_{B_1(\frac{x_n}{\lambda_n})} |y|^{2\kappa/2^*_{s,\kappa}} \frac{|\tilde{u}_n|^2}{|y|^{2\kappa/2^*_{s,\kappa}}} \mathrm{d}y \\ &\le \left(\int_{B_1(\frac{x_n}{\lambda_n})} |y|^{\frac{\kappa(N-2s)}{2s-\kappa}} \mathrm{d}y \right)^{\frac{2s-\kappa}{N-\kappa}} \left(\int_{B_1(\frac{x_n}{\lambda_n})} \frac{|\tilde{u}_n|^{2^*_{s,\kappa}}}{|y|^\kappa} \mathrm{d}y \right)^{\frac{2}{2^*_{s,\kappa}}}. \end{aligned}$$

By the rearrangement inequality, see [17, Theorem 3.4], we have

$$\int_{B_1(\frac{x_n}{\lambda_n})} |y|^{\frac{\kappa(N-2s)}{2s-\kappa}} \mathrm{d}y \leq \int_{B_1(0)} |y|^{\frac{\kappa(N-2s)}{2s-\kappa}} \mathrm{d}y \leq C.$$

Therefore,

$$\int_{B_1(\frac{x_n}{\lambda_n})} \frac{|\tilde{u}_n(y)|^{2^*_{s,\kappa}}}{|y|^{\kappa}} \mathrm{d}y \geq C > 0. \tag{3.9}$$

Now, suppose on the contrary, that $|\frac{x_n}{\lambda_n}| \to \infty$ as $n \to \infty$. Then, for any $y \in B_1(\frac{x_n}{\lambda_n})$, we have $|y| \geq |\frac{x_n}{\lambda_n}| - 1$ for n large. Thus by the Hölder inequality, it follows that

$$\begin{aligned}
\int_{B_1(\frac{x_n}{\lambda_n})} \frac{|\tilde{u}_n(y)|^{2^*_{s,\kappa}}}{|y|^{\kappa}} \mathrm{d}y &\leq \frac{1}{(|\frac{x_n}{\lambda_n}| - 1)^{\kappa}} \int_{B_1(\frac{x_n}{\lambda_n})} |\tilde{u}_n(y)|^{2^*_{s,\kappa}} \mathrm{d}y \\
&\leq \frac{1}{(|\frac{x_n}{\lambda_n}| - 1)^{\kappa}} \left(\int_{B_1(\frac{x_n}{\lambda_n})} |\tilde{u}_n(y)|^{2^*_s} \mathrm{d}y \right)^{\frac{N-\kappa}{N}} \leq \frac{1}{(|\frac{x_n}{\lambda_n}| - 1)^{\kappa}} \|\tilde{u}_n\|^{\frac{N-\kappa}{N}}_{\dot{H}^s(\mathbb{R}^N)} \\
&\leq \frac{C}{(|\frac{x_n}{\lambda_n}| - 1)^{\kappa}} \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

which contradicts (3.9). Hence, $\{\frac{x_n}{\lambda_n}\}_n$ is uniformly bounded, and there exists $R > 0$ such that

$$\int_{B_R(0)} |\tilde{u}_n(y)|^2 \mathrm{d}y \geq \int_{B_1(\frac{x_n}{\lambda_n})} |\tilde{u}_n(y)|^2 \mathrm{d}y \geq C_1 > 0.$$

The compact Sobolev embedding $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^2_{\mathrm{loc}}(\mathbb{R}^N)$ implies that there exists $\tilde{u}$ satisfing

$$\int_{B_R(0)} |\tilde{u}(y)|^2 \mathrm{d}y \geq C_1 > 0,$$

which means $\tilde{u} \not\equiv 0$. Similarly we can get $\tilde{v} \not\equiv 0$. By a Brézis–Lieb-type lemma, see [19, Lemma 2.4], we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} (\mathcal{I}_a * |\tilde{u}_n - \tilde{u}|^{2^\sharp_{h,a}}) |\tilde{u}_n - \tilde{u}|^{2^\sharp_{h,a}} \mathrm{d}x + \int_{\mathbb{R}^N} (\mathcal{I}_a * |\tilde{u}|^{2^\sharp_{h,a}}) |\tilde{u}|^{2^\sharp_{h,a}} \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^N} (\mathcal{I}_a * |\tilde{u}_n|^{2^\sharp_{h,a}}) |\tilde{u}_n|^{2^\sharp_{h,a}} \mathrm{d}x + o(1),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N} (\mathcal{I}_a * |\tilde{v}_n - \tilde{v}|^{2^\sharp_{h,a}}) |\tilde{v}_n - \tilde{v}|^{2^\sharp_{h,a}} \mathrm{d}x + \int_{\mathbb{R}^N} (\mathcal{I}_a * |\tilde{v}|^{2^\sharp_{h,a}}) |\tilde{v}|^{2^\sharp_{h,a}} \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^N} (\mathcal{I}_a * |\tilde{v}_n|^{2^\sharp_{h,a}}) |\tilde{v}_n|^{2^\sharp_{h,a}} \mathrm{d}x + o(1).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
S_{s,a} &= \int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} \tilde{u}_n|^2 - \mu \frac{|\tilde{u}_n|^2}{|x|^{2s}} + |(-\Delta)^{\frac{s}{2}} \tilde{v}_n|^2 - \mu \frac{|\tilde{v}_n|^2}{|x|^{2s}} \right] \mathrm{d}x + o(1) \\
&= \int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} (\tilde{u}_n - \tilde{u})|^2 - \mu \frac{|\tilde{u}_n - \tilde{u}|^2}{|x|^{2s}} + |(-\Delta)^{\frac{s}{2}} |\tilde{v}_n - \tilde{v}|^2 - \mu \frac{|\tilde{v}_n - \tilde{v}|^2}{|x|^{2s}} \right] \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} \tilde{u}|^2 - \mu \frac{|\tilde{u}|^2}{|x|^{2s}} + |(-\Delta)^{\frac{s}{2}} |\tilde{v}|^2 - \mu \frac{|\tilde{v}|^2}{|x|^{2s}} \right] \mathrm{d}x + o(1) \\
&\geq S_{s,a} \left(\int_{\mathbb{R}^N} \left(\mathcal{I}_a * |\tilde{u}_n - \tilde{u}|^{2^\sharp_{h,a}} \right) |\tilde{u}_n - \tilde{u}|^{2^\sharp_{h,a}} \mathrm{d}x + \int_{\mathbb{R}^N} \left(\mathcal{I}_a * |\tilde{v}_n - \tilde{v}|^{2^\sharp_{h,a}} \right) |\tilde{v}_n - \tilde{v}|^{2^\sharp_{h,a}} \right)^{\frac{1}{2^\sharp_{h,a}}} \\
&\quad + S_{s,a} \left(\int_{\mathbb{R}^N} \left(\mathcal{I}_a * |\tilde{u}|^{2^\sharp_{h,a}} \right) |\tilde{u}|^{2^\sharp_{h,a}} \mathrm{d}x + \int_{\mathbb{R}^N} \left(\mathcal{I}_a * |\tilde{v}|^{2^\sharp_{h,a}} \right) |\tilde{v}|^{2^\sharp_{h,a}} \mathrm{d}x \right)^{\frac{1}{2^\sharp_{h,a}}} + o(1) \\
&\geq S_{s,a} \Bigg[\int_{\mathbb{R}^N} \left[\left(\mathcal{I}_a * |\tilde{u}_n - \tilde{u}|^{2^\sharp_{h,a}} \right) |\tilde{u}_n - \tilde{u}|^{2^\sharp_{h,a}} + \left(\mathcal{I}_a * |\tilde{v}_n - \tilde{v}|^{2^\sharp_{h,a}} \right) |\tilde{v}_n - \tilde{v}|^{2^\sharp_{h,a}} \right] \mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^N} \left[\left(\mathcal{I}_a * |\tilde{u}|^{2^\sharp_{h,a}} \right) |\tilde{u}|^{2^\sharp_{h,a}} + \left(\mathcal{I}_a * |\tilde{v}|^{2^\sharp_{h,a}} \right) |\tilde{v}|^{2^\sharp_{h,a}} \right] \mathrm{d}x \Bigg]^{\frac{1}{2^\sharp_{h,a}}} + o(1) \\
&= S_{s,a}.
\end{aligned}
$$

Since $\tilde{u}, \tilde{v} \not\equiv 0$, we obtain

$$
S_{s,a} = \int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} \tilde{u}|^2 - \mu \frac{|\tilde{u}|^2}{|x|^{2s}} + |(-\Delta)^{\frac{s}{2}} \tilde{v}|^2 - \mu \frac{|\tilde{v}|^2}{|x|^{2s}} \right] \mathrm{d}x,
$$

and

$$
\int_{\mathbb{R}^N} \left[(\mathcal{I}_a * |\tilde{u}|^{2^\sharp_{h,a}}) |\tilde{u}|^{2^\sharp_{h,a}} + (\mathcal{I}_a * |\tilde{v}|^{2^\sharp_{h,a}}) |\tilde{v}|^{2^\sharp_{h,a}} \right] \mathrm{d}x = 1.
$$

Hence, $S_{s,a}$ is achieved.

Let $(\tilde{u}, \tilde{v})$ be a minimizer. By inequality (A.11) in [26], we get

$$
\int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} |\tilde{u}||^2 + |(-\Delta)^{\frac{s}{2}} |\tilde{v}||^2 \right] \mathrm{d}x \leq \int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} \tilde{u}|^2 + |(-\Delta)^{\frac{s}{2}} \tilde{v}|^2 \right] \mathrm{d}x,
$$

which implies that $(|\tilde{u}|, |\tilde{v}|)$ is also a minimizer of $S_{s,a}$ and hence $\tilde{u} \geq 0, \tilde{v} \geq 0$. All argument of rearrangement (see [11, 26]) shows that $(\tilde{u}, \tilde{v})$ is radially symmetric. The proof is therefore complete. □

For any $\alpha, \beta > 1$ with $\alpha + \beta = 2^*_{s,b}$, and $0 < \mu < \mu_*$, we define the following best Hardy–Sobolev-type constant:

$$
\Lambda_{s,b} := \inf_{u \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}} u|^2 - \mu \frac{|u|^2}{|x|^{2s}} \right) \mathrm{d}x}{\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{s,b}}}{|x|^b} \mathrm{d}x \right)^{\frac{2}{2^*_{s,b}}}}. \tag{3.10}
$$

We may prove, as in [32] with minor changes, that $\Lambda_{s,b}$ is achieved by a radially symmetric nonnegative function. From this and the definition of $S_{s,b}$, we can get the following relation between $\Lambda_{s,b}$ and $S_{s,b}$.

Theorem 3.2. *$S_{s,b} = f(\tau_{\min})\Lambda_{s,b}$. Here*

$$f(\tau) := \frac{1+\tau^2}{(1+\eta\tau^\beta+\tau^{\alpha+\beta})^{\frac{2}{\alpha+\beta}}}, \qquad \tau \geq 0, \tag{3.11}$$

$$f(\tau_{\min}) := \min_{\tau\geq 0} f(\tau) > 0, \tag{3.12}$$

where $\tau_{\min} \geq 0$ is a minimal point of $f(\tau)$ and therefore a root of the equation

$$2^*_{s,b}\tau^{2^*_{s,b}-2} + \eta\beta\tau^{\beta-2} - \eta\alpha\tau^\beta - 2^*_{s,b} = 0\,, \qquad \tau \geq 0. \tag{3.13}$$

Proof. We mimic the proof of Theorem 1.1 in [10]. By the definition of $f(\tau)$ defined in (3.11), it follows that

$$\lim_{\tau\to 0^+} f(\tau) = \lim_{\tau\to+\infty} f(\tau) = 1.$$

Thus $\min_{\tau\geq 0} f(\tau)$ must be achieved at $\tau_{\min} \geq 0$. Furthermore, direct calculation shows that there exists a positive constant C such that

$$0 < C \leq f(\tau_{\min}) := \min_{\tau>0} f(\tau) \leq 1, \qquad 0 < \tau_{\min} < \infty.$$

From the fact that $f'(\tau_{\min}) = 0$, we deduce that $\tau_{\min}$ is a root of the following equation

$$2^*_{s,b}\tau^{2^*_{s,b}-2} + \eta\beta\tau^{\beta-2} - \eta\alpha\tau^\beta - 2^*_{s,b} = 0, \qquad \tau \geq 0.$$

Suppose that $\{w_n\}_n \subset \dot{H}^s(\mathbb{R}^N)$ is a minimizing sequence for $\Lambda_{s,b}$. Let $\tau_1, \tau_2 > 0$ to be chosen later. Taking $u_n = \tau_1 w_n$ and $v_n = \tau_2 w_n$ in (3.8), we have

$$\frac{\tau_1^2+\tau_2^2}{\left(\tau_1^{2^*_{s,b}} + \tau_2^{2^*_{s,b}} + \eta\tau_1^\alpha\tau_2^\beta\right)^{\frac{2}{2^*_{s,b}}}} \cdot \frac{\|w_n\|^2_{\dot{H}^s}}{\left(\int_{\mathbb{R}^N} \frac{|w_n|^{2^*_{s,b}}}{|x|^b} \mathrm{d}x\right)^{\frac{2}{2^*_{s,b}}}} \geq S_{s,b}. \tag{3.14}$$

Note that

$$f\left(\frac{\tau_2}{\tau_1}\right) = \frac{\tau_1^2+\tau_2^2}{\left(\tau_1^{\alpha+\beta} + \eta\tau_1^\alpha\cdot\tau_2^\beta + \tau_2^{2^*_{s,b}}\right)^{\frac{2}{2^*_{s,b}}}}.$$

Choose τ_1 and τ_2 in (3.14) such that $\frac{\tau_2}{\tau_1} = \tau_{\min}$. Passing to the limit as $n \to \infty$ we have

$$f(\tau_{\min})\Lambda_{s,b} \geq S_{s,b}. \tag{3.15}$$

On the other hand, let $\{(u_n, v_n)\}_n$ be a minimizing sequence of $S_{s,b}$ and define $z_n = \tau_n v_n$, where

$$\tau_n^{\alpha+\beta} = \frac{\int_{\mathbb{R}^N} \frac{|u_n|^{\alpha+\beta}}{|x|^b}\mathrm{d}x}{\int_{\mathbb{R}^N} \frac{|v_n|^{\alpha+\beta}}{|x|^b}\mathrm{d}x}.$$

Then

$$\int_{\mathbb{R}^N} \frac{|z_n|^{\alpha+\beta}}{|x|^b}\mathrm{d}x = \int_{\mathbb{R}^N} \frac{|u_n|^{\alpha+\beta}}{|x|^b}\mathrm{d}x. \tag{3.16}$$

From the Young inequality it follows that

$$\int_{\mathbb{R}^N} \frac{|u_n|^\alpha\cdot|z_n|^\beta}{|x|^b}\mathrm{d}x \leq \frac{\alpha}{\alpha+\beta}\int_{\mathbb{R}^N} \frac{|u_n|^{\alpha+\beta}}{|x|^b}\mathrm{d}x + \frac{\beta}{\alpha+\beta}\int_{\mathbb{R}^N} \frac{|z_n|^{\alpha+\beta}}{|x|^b}\mathrm{d}x.$$

Thus by (3.16) we have

$$\int_{\mathbb{R}^N} \frac{|u_n|^\alpha \cdot |z_n|^\beta}{|x|^b} \mathrm{d}x \leq \int_{\mathbb{R}^N} \frac{|u_n|^{\alpha+\beta}}{|x|^b} \mathrm{d}x = \int_{\mathbb{R}^N} \frac{|z_n|^{\alpha+\beta}}{|x|^b} \mathrm{d}x. \tag{3.17}$$

Consequently,

$$\begin{aligned}
\frac{\|u_n\|^2_{\dot{H}^s} + \|v_n\|^2_{\dot{H}^s}}{\left(\int_{\mathbb{R}^N} \frac{|u_n|^{2^*_{s,b}} + |v_n|^{2^*_{s,b}} + \eta |u_n|^\alpha |v_n|^\beta}{|x|^b} \mathrm{d}x\right)^{\frac{2}{2^*_{s,b}}}} &= \frac{\|u_n\|^2_{\dot{H}^s} + \|v_n\|^2_{\dot{H}^s}}{\left[\left(1+\eta\tau_n^{-\beta} + \tau_n^{-(\alpha+\beta)}\right) \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_{s,b}}}{|x|^b} \mathrm{d}x\right]^{\frac{2}{2^*_{s,b}}}} \\
&= \frac{\|u_n\|^2_{\dot{H}^s}}{\left[\left(1+\eta\tau_n^{-\beta} + \tau_n^{-(\alpha+\beta)}\right) \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_{s,b}}}{|x|^b} \mathrm{d}x\right]^{\frac{2}{2^*_{s,b}}}} + \frac{\tau_n^{-2}\|z_n\|^2_{\dot{H}^s}}{\left[\left(1+\eta\tau_n^{-\beta} + \tau_n^{-(\alpha+\beta)}\right) \int_{\mathbb{R}^N} \frac{|z_n|^{2^*_{s,b}}}{|x|^b} \mathrm{d}x\right]^{\frac{2}{2^*_{s,b}}}} \\
&\geq f(\tau_n^{-1})\Lambda_{s,b} \geq f(\tau_{\min})\Lambda_{s,b}.
\end{aligned}$$

As $n \to \infty$, we have

$$S_{s,b} \geq f(\tau_{\min})\Lambda_{s,b}. \tag{3.18}$$

From (3.15) and (3.18) it follows that

$$S_{s,b} = f(\tau_{\min})\Lambda_{s,b}. \tag{3.19}$$

Thus, the proof is complete. □

4 Proof of Theorem 1.2

In this section, we investigate the existence of solutions for problem (1.1). We first give some technical lemmas so that we can use the mountain pass lemma to seek critical points of problem (1.1).

The Nehari manifold related to I is given by

$$\mathcal{N} = \left\{(u,v) \subset \dot{H}^s(\mathbb{R}^N)\backslash\{0\} \times \dot{H}^s(\mathbb{R}^N)\backslash\{0\} : \langle I'(u,v),(u,v)\rangle = 0\right\}.$$

Then a minimizer of the minimization problem

$$c_0 = \inf_{u\in\mathcal{N}} I(u,v)$$

is a solution of problem (1.1). In order to establish the existence of solutions for problem (1.1), we set

$$c_\gamma = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], \dot{H}^s(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(e) < 0\}$ and

$$c_s = \inf_{(u,v)\in H} \max_{t\geq 0} I(t(u,v)),$$

and

$$c^* := \min\left\{\frac{a+2s}{2(N+a)} S_{s,a}^{\frac{N+a}{a+2s}}, \frac{2s-b}{2(N-b)} S_{s,b}^{\frac{N-b}{2s-b}}\right\}.$$

Then we have the following result.

Lemma 4.1. *Suppose $0 < a, b < 2s < N$. Then we have*

$$c_0 = c_\gamma = c_s. \tag{4.1}$$

The proof of Lemma 4.1 is standard. The further details with minor changes can be derived as in the proof of [29, Theorem 4.2].

However, the following lemma reveals that $c_s < c^*$.

Lemma 4.2. *If $0 < a, b < 2s < 2N$, then $c_s < c^*$.*

Proof. By Lemma 3.1, there exist minimizers $(u_1, v_1) \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\} \times \dot{H}^s(\mathbb{R}^N) \setminus \{0\}$ of $S_{s,a}$ and $(u_2, v_2) \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\} \times \dot{H}^s(\mathbb{R}^N) \setminus \{0\}$ of $S_{s,b}$. For $t \geq 0$, we define

$$\begin{aligned} f_a(t) = {} & \frac{1}{2}t^2 \int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} u_1|^2 - \mu \frac{|u_1|^2}{|x|^{2s}} + |(-\Delta)^{\frac{s}{2}} v_1|^2 - \mu \frac{|v_1|^2}{|x|^{2s}} \right] \mathrm{d}x \\ & - \frac{t^{2\cdot 2^\sharp_{h,a}}}{2 \cdot 2^\sharp_{h,a}} \iint_{\mathbb{R}^{2N}} \frac{|u_1(x)|^{2^\sharp_{h,a}} |u_1(y)|^{2^\sharp_{h,a}} + |v_1(x)|^{2^\sharp_{h,a}} |v_1(y)|^{2^\sharp_{h,a}}}{|x-y|^{N-a}} \mathrm{d}x\mathrm{d}y, \end{aligned}$$

and

$$\begin{aligned} f_b(t) = {} & \frac{1}{2}t^2 \int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} u_2|^2 - \mu \frac{|u_2|^2}{|x|^{2s}} + |(-\Delta)^{\frac{s}{2}} v_2|^2 - \mu \frac{|v_2|^2}{|x|^{2s}} \right] \mathrm{d}x \\ & - \frac{t^{2^*_{s,b}}}{2^*_{s,b}} \int_{\mathbb{R}^N} \frac{|u_2|^{2^*_{s,b}} + |v_2|^{2^*_{s,b}} + \eta |u|^\alpha |v_2|^\beta}{|x|^b} \mathrm{d}x. \end{aligned}$$

Obviously,

$$\max_{t \geq 0} I(t(u_1, v_1)) \leq \max_{t \geq 0} f_a(t),$$

and

$$\max_{t \geq 0} I(t(u_2, v_2) \leq \max_{t \geq 0} f_b(t).$$

We may verify that the function $f_a(t)$ attains its maximum at

$$t_a = \left(\frac{\int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} u_1|^2 - \mu \frac{|u_1|^2}{|x|^{2s}} + |(-\Delta)^{\frac{s}{2}} v_1|^2 - \mu \frac{|v_1|^2}{|x|^{2s}} \right] \mathrm{d}x}{\iint_{\mathbb{R}^{2N}} \frac{|u_1(x)|^{2^\sharp_{h,a}} |u_1(y)|^{2^\sharp_{h,a}} + |v_1(x)|^{2^\sharp_{h,a}} |v_1(y)|^{2^\sharp_{h,a}}}{|x-y|^{N-a}} \mathrm{d}x\mathrm{d}y} \right)^{\frac{1}{2 \cdot 2^\sharp_{h,a} - 2}},$$

and the function $f_b(t)$ attains its maximum at

$$t_b = \left(\frac{\int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} u_2|^2 - \mu \frac{|u_2|^2}{|x|^{2s}} + |(-\Delta)^{\frac{s}{2}} v_2|^2 - \mu \frac{|v_2|^2}{|x|^{2s}} \right] \mathrm{d}x}{\int_{\mathbb{R}^N} \frac{|u_2|^{2^*_{s,b}} + |v_2|^{2^*_{s,b}} + \eta |u_2|^\alpha |v_2|^\beta}{|x|^b} \mathrm{d}x} \right)^{\frac{1}{2^*_{s,b} - 2}}.$$

Thus it yields that

$$\max_{t \geq 0} f_a(t) = f_a(t_a) = \frac{a+2s}{2(N+a)} S_{s,a}^{\frac{N+a}{a+2s}},$$

and

$$\max_{t \geq 0} f_b(t) = f_b(t_b) = \frac{2s-b}{2(N-b)} S_{s,b}^{\frac{N-b}{2s-b}}.$$

Now we show

$$\max_{t \geq 0} I(t(u_1, v_1)) < f_a(t_a),$$

and

$$\max_{t\geq 0} I(t(u_2,v_2)) < f_b(t_b).$$

Indeed, if there exist $t_1>0, t_2>0$ such that $I(t_1(u_1,v_1)) = f_a(t_a)$, $I(t_2(u_2,v_2)) = f_b(t_b)$, that is,

$$f_a(t_1) - \frac{t^{2^*_{s,b}}}{2^*_{s,b}} \int_{\mathbb{R}^N} \frac{|u_1|^{2^*_{s,b}} + |v_1|^{2^*_{s,b}} + \eta |u_1|^{\alpha}|v_1|^{\beta}}{|x|^b} \mathrm{d}x = f_a(t_a),$$

and

$$f_b(t_2) - \frac{t_2^{2\cdot 2^\sharp_{h,a}}}{2\cdot 2^\sharp_{h,a}} \iint_{\mathbb{R}^{2N}} \frac{|u_2(x)|^{2^\sharp_{h,a}}|u_2(y)|^{2^\sharp_{h,a}} + |v_2(x)|^{2^\sharp_{h,a}}|v_2(y)|^{2^\sharp_{h,a}}}{|x-y|^{N-a}} \mathrm{d}x\mathrm{d}y = f_b(t_b).$$

which yields a contradiction and hence the assertion follows. □

Next we show that the functional I satisfies the geometrical conditions in the mountain pass lemma without the (PS) condition in [2].

Lemma 4.3. *Suppose $0<a,b<2s<N$.*

- (i) *There exist positive numbers ρ and α_0 such that $I(u,v)|_{\|(u,v)\|_H=\rho} \geq \alpha_0$ for all $u,v \in \dot{H}^s(\mathbb{R}^N)$.*
- (ii) *There exists $e \in H$ with $\|e\| > \rho$ such that $I(e) < 0$.*

Proof. Let us show (i). From (2.4) and Proposition 1.1, it is simple to see that

$$\begin{aligned} I(u,v) &= \frac{1}{2}\|(u,v)\|^2 - \frac{1}{2^*_{s,b}} \int_{\mathbb{R}^N} \frac{|u|^{2^*_{s,b}} + |v|^{2^*_{s,b}} + \eta|u|^{\alpha}|v|^{\beta}}{|x|^b}\mathrm{d}x \\ &\quad - \frac{1}{2\cdot 2^\sharp_{h,a}} \iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{2^\sharp_{h,a}}|u(y)|^{2^\sharp_{h,a}} + |v(x)|^{2^\sharp_{h,a}}|v(y)|^{2^\sharp_{h,a}}}{|x-y|^{N-a}}\mathrm{d}x\mathrm{d}y \\ &\geq \frac{1}{2}\left(\|u\|^2_{\dot{H}^s(\mathbb{R}^N)} + \|v\|^2_{\dot{H}^s(\mathbb{R}^N)}\right) - \frac{C_1}{2\cdot 2^\sharp_{h,a}}\left(\|u\|^{22^\sharp_{h,a}}_{\dot{H}^s(\mathbb{R}^N)} + \|v\|^{22^\sharp_{h,a}}_{\dot{H}^s(\mathbb{R}^N)}\right) \\ &\quad - \frac{C_2}{2^*_{s,b}}\left(\|u\|^{2^*_{s,b}\theta}_{\dot{H}^s(\mathbb{R}^N)} + \|v\|^{2^*_{s,b}\theta}_{\dot{H}^s(\mathbb{R}^N)}\right) \\ &\geq \frac{1}{2}\left(\|u\|^2_{\dot{H}^s(\mathbb{R}^N)} + \|v\|^2_{\dot{H}^s(\mathbb{R}^N)}\right) - \frac{C_3}{2\cdot 2^\sharp_{h,a}}\left(\|u\|^2_{\dot{H}^s(\mathbb{R}^N)} + \|v\|^2_{\dot{H}^s(\mathbb{R}^N)}\right)^{2^\sharp_{h,a}} \\ &\quad - \frac{C_4}{2^*_{s,b}}\left(\|u\|^2_{\dot{H}^s(\mathbb{R}^N)} + \|v\|^2_{\dot{H}^s(\mathbb{R}^N)}\right)^{\frac{2^*_{s,b}\theta}{2}} \\ &\geq \alpha_0 > 0 \end{aligned}$$

for $\|(u,v)\| = \rho > 0$ small. The first assertion is proved.

For (ii), notice that

$$\begin{aligned} I(t(u_0,v_0)) &= \frac{t^2}{2}\|(u_0,v_0)\|^2 - \frac{t^{2^*_{s,b}}}{2^*_{s,b}} \int_{\mathbb{R}^N} \frac{|u_0|^{2^*_{s,b}} + |v_0|^{2^*_{s,b}} + \eta|u_0|^{\alpha}|v_0|^{\beta}}{|x|^b}\mathrm{d}x \\ &\quad - \frac{t^{2\cdot 2^\sharp_{h,a}}}{2\cdot 2^\sharp_{h,a}} \iint_{\mathbb{R}^{2N}} \frac{|u_0(x)|^{2^\sharp_{h,a}}|u_0(y)|^{2^\sharp_{h,a}} + |v_0(x)|^{2^\sharp_{h,a}}|v_0(y)|^{2^\sharp_{h,a}}}{|x-y|^{N-a}}\mathrm{d}x\mathrm{d}y \\ &\to -\infty \quad \text{as } t\to\infty. \end{aligned}$$

Taking t_0 large enough such that $\|t_0(u_0, v_0)\| > \rho$ and $I(t_0(u_0, v_0)) < 0$. Put $e = t_0(u_0, v_0)$, then the assertion (ii) follows. $\square$

The following result implies the non-vanishing of $(PS)_c$ sequences.

Lemma 4.4. *Suppose $0 < a, b < 2s < N$. Let $\{(u_n, v_n)\} \subset H$ be a $(PS)_c$ sequence of I with $c \in (0, c^*)$. Then*

$$\int_{\mathbb{R}^N} \frac{|u|^{2^*_{s,b}} + |v|^{2^*_{s,b}} + \eta |u|^\alpha |v|^\beta}{|x|^b} \mathrm{d}x > 0, \tag{4.2}$$

and

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{2^\sharp_{h,a}} |u(y)|^{2^\sharp_{h,a}} + |v(x)|^{2^\sharp_{h,a}} |v(y)|^{2^\sharp_{h,a}}}{|x-y|^{N-a}} \mathrm{d}x\mathrm{d}y > 0. \tag{4.3}$$

Proof. Since

$$I(u_n, v_n) \to c, \quad I'(u_n, v_n) \to 0$$

as $n \to \infty$, in particular,

$$\begin{aligned} o(1) + \langle I'(u_n, v_n), (u_n, v_n) \rangle = \|(u_n, v_n)\|^2 + \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_{s,b}} + |v_n|^{2^*_{s,b}} + \eta |u_n|^\alpha |v_n|^\beta}{|x|^b} \mathrm{d}x \\ - \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)|^{2^\sharp_{h,a}} |u_n(y)|^{2^\sharp_{h,a}} + |v_n(x)|^{2^\sharp_{h,a}} |v_n(y)|^{2^\sharp_{h,a}}}{|x-y|^{N-a}} \mathrm{d}x\mathrm{d}y, \end{aligned}$$

we get

$$\begin{aligned} c + o(1)\|(u_n, v_n)\| &= I(u_n, v_n) - \frac{1}{2^*_{s,b}} \langle I'(u_n, v_n), (u_n, v_n) \rangle \\ &\geq \left(\frac{1}{2} - \frac{1}{2^*_{s,b}} \right) \|(u_n, v_n)\|^2, \end{aligned}$$

which implies that $\{(u_n, v_n)\}_n$ is uniformly bounded in H.

Now we give the proof by contradiction. Suppose

$$\int_{\mathbb{R}^N} \frac{|u_n|^{2^*_{s,b}} + |v_n|^{2^*_{s,b}} + \eta |u_n|^\alpha |v_n|^\beta}{|x|^b} \mathrm{d}x = 0,$$

then, from $I(u_n, v_n) \to c$ and $I'(u_n, v_n) \to 0$, we have

$$c + o(1) = \frac{1}{2}\|(u_n, v_n)\|^2 - \frac{1}{2 \cdot 2^\sharp_{h,a}} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)|^{2^\sharp_{h,a}} |u_n(y)|^{2^\sharp_{h,a}} + |v_n(x)|^{2^\sharp_{h,a}} |v_n(y)|^{2^\sharp_{h,a}}}{|x-y|^{N-a}} \mathrm{d}x\mathrm{d}y,$$

and

$$o(1) = \|(u_n, v_n)\|^2 - \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)|^{2^\sharp_{h,a}} |u_n(y)|^{2^\sharp_{h,a}} + |v_n(x)|^{2^\sharp_{h,a}} |v_n(y)|^{2^\sharp_{h,a}}}{|x-y|^{N-a}} \mathrm{d}x\mathrm{d}y. \tag{4.4}$$

Thus we get

$$c + o(1) = \left(\frac{1}{2} - \frac{1}{2 \cdot 2^\sharp_{h,a}} \right) \|(u_n, v_n)\|^2. \tag{4.5}$$

Moreover, by (4.4) and the definition of $S_{s,a}$, we have

$$\|(u_n, v_n)\|^2 \geq S_{s,a} \|(u_n, v_n)\|^{\frac{2}{2^\sharp_{h,a}}} \Rightarrow \|(u_n, v_n)\|^2 \geq S_{s,a}^{\frac{N+a}{a+2s}}.$$

Therefore, from (4.5) it follows that

$$c \geq \frac{a+2s}{2(N+a)} S_{s,a}^{\frac{N+a}{a+2s}},$$

which is a contradiction. Similarly, we can prove (4.3). Hence the proof is complete. □

We are now ready to prove the existence of solutions for problem (1.1).

Proof of Theorem 1.2. By Lemma 4.3, there exists a $(PS)_{c_s}$ sequence $\{(u_n, v_n)\}_n$, which is bounded in H. Thus we may assume that

$$\begin{aligned} &u_n \rightharpoonup u,\ v_n \rightharpoonup v && \text{weakly in } \dot{H}^s(\mathbb{R}^N)\,, \\ &u_n \to u,\ v_n \to v && \text{a.e. in } \mathbb{R}^N\,, \\ &u_n \to u,\ v_n \to v && \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^N) \text{ for all } 1 \leq p < 2_s^*. \end{aligned}$$

We claim that (u, v) is a nontrivial solution of (1.1). Indeed, by Lemma 4.2 and Lemma 4.4, we claim that (4.2) and (4.3) hold true. Hence, by (2.5), there exists $C > 0$ such that

$$\|u_n\|_{\mathcal{L}^{2,N-2s}(\mathbb{R}^N)} \geq C, \qquad \|v_n\|_{\mathcal{L}^{2,N-2s}(\mathbb{R}^N)} \geq C.$$

So we can find $\lambda_n > 0$ and $x_n \in \mathbb{R}^N$ such that

$$\lambda_n^{-2s} \int_{B_{\lambda_n}(x_n)} |u_n|^2 \mathrm{d}y \geq \|u_n\|^2_{\mathcal{L}^{2,N-2s}(\mathbb{R}^N)} - \frac{C}{2n} \geq C_1 > 0. \tag{4.6}$$

Let $\tilde{u}_n(x) = \lambda_n^{\frac{N-2s}{2}} u_n(x_n + \lambda_n x), \tilde{v}_n(x) = \lambda_n^{\frac{N-2s}{2}} v_n(x_n + \lambda_n x)$. We may readily verify that as $n \to \infty$

$$\tilde{I}(\tilde{u}_n, \tilde{v}_n) = I(u_n, v_n) \to c_s, \qquad |\langle \tilde{I}'(\tilde{u}_n, \tilde{v}_n), (\varphi_1, \varphi_2)\rangle \to 0$$

for all $(\varphi_1, \varphi_2) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$, where

$$\begin{aligned} \tilde{I}(\tilde{u}_n, \tilde{v}_n) = &\frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \tilde{u}_n|^2 \mathrm{d}x - \frac{1}{2}\mu \int_{\mathbb{R}^N} \frac{|\tilde{u}_n|^2}{|x + \frac{x_n}{\lambda_n}|^{2s}} \mathrm{d}x \\ &+ \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \tilde{v}_n|^2 \mathrm{d}x - \frac{1}{2}\mu \int_{\mathbb{R}^N} \frac{|\tilde{v}_n|^2}{|x + \frac{x_n}{\lambda_n}|^{2s}} \mathrm{d}x \\ &- \frac{1}{2 \cdot 2^\sharp_{h,a}} \iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}_n(x)|^{2^\sharp_{h,a}} |\tilde{u}_n(y)|^{2^\sharp_{h,a}} + |\tilde{v}_n(x)|^{2^\sharp_{h,a}} |\tilde{v}_n(y)|^{2^\sharp_{h,a}}}{|x-y|^{N-a}} \mathrm{d}x\mathrm{d}y \\ &- \frac{1}{2^*_{s,b}} \int_{\mathbb{R}^N} \frac{|\tilde{u}_n|^{2^*_{s,b}} + |\tilde{v}_n|^{2^*_{s,b}} + \eta |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta}{|x + \frac{x_n}{\lambda_n}|^b} \mathrm{d}x. \end{aligned}$$

Then by the same processes in Lemma 4.4, we get that $\{(u_n, v_n)\}_n$ is uniformly bounded in H. Thus there exists $\{(\tilde{u}, \tilde{v})\}$ such that

$$\begin{aligned} &\tilde{u}_n \rightharpoonup \tilde{u},\ \tilde{v}_n \rightharpoonup \tilde{v} && \text{weakly in } \dot{H}^s(\mathbb{R}^N)\,, \\ &\tilde{u}_n \to u,\ v_n \to \tilde{v} && \text{a.e. in } \mathbb{R}^N\,, \\ &\tilde{u}_n \to \tilde{u},\ \tilde{v}_n \to \tilde{v} && \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^N) \text{ for all } 1 \leq p < 2_s^*, \end{aligned}$$

and by

$$\int_{B_R(0)} |\tilde{u_n}(y)|^2 \mathrm{d}y \geq C_1 > 0, \qquad \int_{B_R(0)} |\tilde{v}_n(y)|^2 \mathrm{d}y \geq C_1 > 0,$$

it yields that $\tilde{u} \not\equiv 0, \tilde{v} \not\equiv 0$. We claim that $\{\frac{x_n}{\lambda_n}\}_n$ is bounded. Indeed, if $\frac{x_n}{\lambda_n} \to \infty$, then for any $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \frac{\tilde{u}_n\varphi_1 + \tilde{v}_n\varphi_2}{|x + \frac{x_n}{\lambda_n}|^{2s}} \mathrm{d}x \to 0,$$

$$\int_{\mathbb{R}^N} \frac{|\tilde{u}_n|^{2^*_{s,b}-2}\tilde{u}_n\varphi_1 + |\tilde{v}_n|^{2^*_{s,b}-2}\tilde{v}_n\varphi_2 + \eta(\alpha|\tilde{u}_n|^{\alpha-2}\tilde{u}_n\varphi_1|\tilde{v}_n|^\beta + \beta|\tilde{u}_n|^\alpha|\tilde{v}_n|^{\beta-2}\tilde{v}_n\varphi_2)}{|x + \frac{x_n}{\lambda_n}|^b} \mathrm{d}x \to 0$$

as $n \to \infty$. Hence $(\tilde{u}, \tilde{v})$ solves the equation

$$(-\Delta)^s\tilde{u} - \mu\frac{\tilde{u}}{|x|^{2s}} + (-\Delta)^s\tilde{v} - \mu\frac{\tilde{v}}{|x|^{2s}} = \left(\mathcal{I}_a * |\tilde{u}|^{2^\sharp_{h,a}}\right)|\tilde{u}|^{2^\sharp_{h,a}-2}\tilde{u} + \left(\mathcal{I}_a * |\tilde{v}|^{2^\sharp_{h,a}}\right)|\tilde{v}|^{2^\sharp_{h,a}-2}\tilde{v} \quad \text{in } \mathbb{R}^N,$$

and we obtain

$$\begin{aligned} c_s &= \tilde{I}(\tilde{u}_n, \tilde{v}_n) - \frac{1}{2}\langle \tilde{I}'(\tilde{u}_n, \tilde{v}_n), (\tilde{u}_n, \tilde{v}_n)\rangle + o(1) \\ &= \left(\frac{1}{2} - \frac{1}{2\cdot 2^\sharp_{h,a}}\right)\iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}_n(x)|^{2^\sharp_{h,a}}|\tilde{u}_n(y)|^{2^\sharp_{h,a}} + |\tilde{v}_n(x)|^{2^\sharp_{h,a}}|\tilde{v}_n(y)|^{2^\sharp_{h,a}}}{|x-y|^{N-a}} \mathrm{d}x\mathrm{d}y \\ &\quad + \left(\frac{1}{2} - \frac{1}{2^*_{s,b}}\right)\int_{\mathbb{R}^N} \frac{|\tilde{u}_n|^{2^*_{s,a}} + |\tilde{v}_n|^{2^*_{s,a}} + \eta|\tilde{u}_n|^\alpha|\tilde{v}_n|^\beta}{|x + \frac{x_n}{\lambda_n}|^b} \mathrm{d}x + o(1) \\ &\geq \left(\frac{1}{2} - \frac{1}{2\cdot 2^\sharp_{h,a}}\right)\iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}_n(x)|^{2^\sharp_{h,a}}|\tilde{u}_n(y)|^{2^\sharp_{h,a}} + |\tilde{v}_n(x)|^{2^\sharp_{h,a}}|\tilde{v}_n(y)|^{2^\sharp_{h,a}}}{|x-y|^{N-a}} \mathrm{d}x\mathrm{d}y + o(1) \\ &= I(\tilde{u}, \tilde{v}) \geq c^*, \end{aligned}$$

which contradicts Lemma 4.2. Let $\hat{u}_n(x) = \lambda_n^{\frac{N-2s}{2}} u_n(\lambda_n x), \hat{v}_n(x) = \lambda_n^{\frac{N-2s}{2}} v_n(\lambda_n x)$. Then we can verify that

$$I(\hat{u}_n, \hat{v}_n) = I(u_n, v_n) \to c_m, \quad I'(\hat{u}_n, \hat{v}_n) \to 0 \quad \text{as } n \to \infty.$$

Arguing as before, we have $\hat{u}_n \rightharpoonup \hat{u}, \hat{v}_n \rightharpoonup \hat{v}$ in $\dot{H}^s(\mathbb{R}^N)$, which implies that $(\hat{u}, \hat{v})$ is a solution of (1.1). Furthermore, there exists $R > 0$ such that

$$\int_{B_R(0)} |\hat{u}_n(y)|^2 \mathrm{d}y = \lambda_n^{-2s}\int_{B_{\lambda_n}(x_n)} |u_n(y)|^2 \mathrm{d}y \geq C_1 > 0, \qquad \int_{B_R(0)} |\hat{v}_n(y)|^2 \mathrm{d}y > 0. \tag{4.7}$$

The compact embedding $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^2_{\mathrm{loc}}(\mathbb{R}^N)$ implies that $u_n \to \hat{u}, v_n \to \hat{v}$ in $L^2(B_R(0))$ and $\hat{u}, \hat{v} \not\equiv 0$. Moreover,

$$c_s \geq I(\hat{u}, \hat{v}) \geq c_0 = c_s.$$

Hence, $(\hat{u}, \hat{v})$ is a nontrivial solution of (1.1) satisfying $I(\hat{u}, \hat{v}) = c_s$. This ends the proof of Theorem 1.2. □

Acknowledgements

The authors express their gratitude to the referee for his/her careful reading and helpful suggestions. Moreover, the authors thank Professor Giovanni Molica Bisci for valuable comments and suggestions on improving this paper. L. Wang was supported by National Natural Science Foundation of China (No. 11561024, No. 11661035), Natural Science Foundation of Jiangxi Province (No. 20151BAB201017) and the science and technology project of Jiangxi provincial education department (No. GJJ150537). B. Zhang was supported by Natural Science Foundation of Heilongjiang Province of China (No. A201306) and Research Foundation of Heilongjiang Educational Committee (No. 12541667).

References

[1] G. Autuori, A. Fiscella, P. Pucci, Stationary Kirchhoff problems involving a fractional elliptic operator and a critical nonlinearity, *Nonlinear Anal.* **125**(2015), 699–714. MR3373607

[2] H. Brézis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* **36**(1983), 437–477. MR0709644

[3] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations* **32**(2007), 1245–1260. MR2354493

[4] L. Caffarelli, Non-local diffusions, drifts and games, in: *Nonlinear partial differential equations*, Abel Symposia No. 7, Springer, Heidelberg, 2012, pp. 37–52. MR3289358

[5] L. Caffarelli, J. L. Vázquez, Asymptotic behaviour of a porous medium equation with fractional diffusion, *Discrete Contin. Dyn. Syst.* **29**(2011), 1393–1404. MR2773189

[6] W. Chen, C. Li, B. Ou, Classification of solutions for an integral equation, *Comm. Pure Appl. Math.* **59**(2006), 330–343. MR2200258

[7] A. Cotsiolis, N. K. Travoularis, Best constants for Sobolev inequalities for higher order fractional derivatives, *J. Math. Anal. Appl.* **295**(2004), 225–236. MR2064421

[8] M. Caponi, P. Pucci, Existence theorems for entire solutions of stationary Kirchhoff fractional p-Laplacian equations, *Ann. Mat. Pura Appl.* **195**(2016), 2099–2129. MR3558321

[9] Z. Deng, Y. Huang, Existence and multiplicity of symmetric solutions for semilinear elliptic equations with singular potentials and critical Hardy–Sobolev exponents, *J. Math. Anal. Appl.* **393**(2012), 273–284. MR2921668

[10] K. Deng, S. Peng, Existence and asymptotic properties of solutions to elliptic systems involving multiple critical exponents, *Sci. China Math.* **54**(2011), 243–256. MR2771201

[11] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.* **229**(2012), 521–573. MR2944369

[12] R. Filippucci, P. Pucci, F. Robert, On a p-Laplace equation with multiple critical nonlinearities, *J. Math. Pures Appl.* **91**(2009), 156–177. MR2498753

[13] A. Fiscella, E. Valdinoci, A critical Kirchhoff type problem involving a nonlocal operator, *Nonlinear Anal.* **94**(2014), 156–170. MR3120682

[14] A. Fiscella, G. Molica Bisci, R. Servadei, Bifurcation and multiplicity results for critical nonlocal fractional Laplacian problems, *Bull. Sci. Math.* **140**(2016), 14–35. MR3446947

[15] A. Fiscella, P. Pucci, S. Saldi, Existence of entire solutions for Schrödinger–Hardy systems involving two fractional operators, *Nonlinear Anal.* **158**(2017), 109–131. MR3661432

[16] N. Ghoussoub, S. Shakerian, Borderline variational problems involving fractional Laplacians and critical singularities, *Adv. Nonlinear Stud.* **15**(2015), 527–555. MR3366777

[17] E. H. Lieb, M. Loss, *Analysis*, Graduate Studies in Mathematics, Vol. 14, American Mathematical Society, Providence, 2001. MR1817225; url

[18] N. Laskin, Fractional Schrödinger equation, *Phys. Rev. E (3)* **66**(2002), 056108, 7 pp. MR1948569

[19] V. Moroz, J. Schaftingen, Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics, *J. Funct. Anal.* **265**(2013), 153–184. MR3056699

[20] G. Molica Bisci, R. Servadei, A Brézis–Nirenberg splitting approach for nonlocal fractional problems, *Nonlinear Anal.* **119**(2015), 341–353. MR3334193

[21] G. Molica Bisci, V. Rădulescu, R. Servadei, *Variational methods for nonlocal fractional problems*, Cambridge University Press, Cambridge, 2016. MR3445279

[22] P. Pucci, S. Saldi, Critical stationary Kirchhoff equations in $\mathbb{R}^N$ involving nonlocal operators, *Rev. Mat. Iberoam.* **32**(2016), 1–22. MR3470662

[23] P. Pucci, R. Servadei, Existence, non-existence and regularity of radial ground states for p-Laplacian equations with singular weights, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **25**(2008), 505–537. MR2422077

[24] P. Pucci, M. Q. Xiang, B. L. Zhang, Multiple solutions for nonhomogeneous Schrödinger–Kirchhoff type equations involving the fractional p-Laplacian in $\mathbb{R}^N$, *Calc. Var. Partial Differetial Equations*, **54**(2015), 2785–2806. MR3412392

[25] P. Pucci, M. Q. Xiang, B. L. Zhang, Existence and multiplicity of entire solutions for fractional p-Kirchhoff equations, *Adv. Nonlinear Anal.* **5**(2016), 27–55. MR3456737

[26] R. Servadei, E. Valdinoci, Variational methods for non-local operators of elliptic type, *Discrete Contin. Dyn. Syst.* **33**(2013), 2105–2137. MR3002745

[27] R. Servadei, E. Valdinoci, Mountain pass solutions for non-local elliptic operators, *J. Math. Anal. Appl.* **389**(2012), 887–898. MR2879266

[28] R. Servadei, E. Valdinoci, On the spectrum of two different fractional operators, *Proc. Roy. Soc. Edinburgh Sect. A* **144**(2014), 831–855. MR3233760

[29] M. Willem, Minimax theorems, Birkhäuser, Boston, 1996. MR1400007; url

[30] M. Xiang, B. Zhang, V. Rădulescu, Multiplicity of solutions for a class of quasilinear Kirchhoff system involving the fractional p-Laplacian, *Nonlinearity* **29**(2016), 3186–3205. MR3551061

[31] M. Xiang, B. Zhang, Z. Wei, Existence of solutions to a class of quasilinear Schrödinger system involving the fractional p-Laplacian, *Electron. J. Qual. Theory Differ. Equ.* **2016**, No. 107, 1–15. MR3582900

[32] J. Yang, Fractional Sobolev–Hardy inequality in $\mathbb{R}^N$, *Nonlinear Anal.* **119**(2015), 179–185. MR3334182

[33] J. Yang, F. Wu, Doubly critical problems involving fractional Laplacians in $\mathbb{R}^N$, *Adv. Nonlinear Stud.*, 2017, published online. url

[34] D. Yafaev, Sharp constants in the Hardy–Rellich inequalities, *J. Funct. Anal.* **168**(1999), 121–144. MR1717839